**Rules for Exponents and Logarithms**

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| **Zero-Exponent Rule:** a0 = 1, this says that anything raised to the zero power is 1. | Zero-Exponent Rule Examples |
| **Power Rule** (Powers to Powers): (am)n = amn, this says that to raise a power to a power you need to multiply the exponents. There are several other rules that go along with the power rule, such as the product-to-powers rule and the quotient-to-powers rule. | Power Rule Examples |
| **Negative Exponent Rule**: Negative Exponent Rule, this says that negative exponents in the numerator get moved to the denominator and become positive exponents. Negative exponents in the denominator get moved to the numerator and become positive exponents. Only move the negative exponents. | Negative Exponents Rule Examples |
| **Product Rule**: am ∙ an = am + n, this says that to multiply two exponents with the same base, you keep the base and add the powers. | Product Rule Examples |
| **Quotient Rule**: Quotient Rule, this says that to divide two exponents with the same base, you keep the base and subtract the powers. This is similar to reducing fractions; when you subtract the powers put the answer in the numerator or denominator depending on where the higher power was located. If the higher power is in the denominator, put the difference in the denominator and vice versa, this will help avoid negative exponents. | Quotient Rule Examples |

Now that we have reviewed the rules for exponents, here are the steps required for simplifying exponential expressions (notice that we apply the rules in the same order the rule were written above):

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| **Step 1**: | Apply the Zero-Exponent Rule. Change anything raised to the zero power into a 1. |
| **Step 2**: | Apply the Power Rule. Multiply (or distribute) the exponent outside the parenthesis with every exponent inside the parenthesis, remember that if there is no exponent shown, then the exponent is 1. |
| **Step 3**: | Apply the Negative Exponent Rule. Negative exponents in the numerator get moved to the denominator and become positive exponents. Negative exponents in the denominator get moved to the numerator and become positive exponents. Only move the negative exponents. Note that the order in which things are moved does not matter. |
| **Step 4**: | Apply the Product Rule. To multiply two exponents with the same base, you keep the base and add the powers. |
| **Step 5**: | Apply the Quotient Rule. This is similar to reducing fractions; when you subtract the powers put the answer in the numerator or denominator depending on where the higher power was located. If the higher power is in the denominator, put the difference in the denominator and vice versa, this will help avoid negative exponents and a repeat of step 3. |
| **Step 6**: | Raise each coefficient (or number) to the appropriate power and then simplify or reduce any remaining fractions. |

**Example 1**– Simplify: Example 1

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| **Step 1**: Apply the Zero-Exponent Rule. In this case, there are no zero powers. | Step 1 |
| **Step 2**: Apply the Power Rule. | Step 2 |
| **Step 3**: Apply the Negative Exponent Rule. Move every negative exponent in the numerator to the denominator and vice versa. | Step 3 |
| **Step 4**: Apply the Product Rule. | Step 4 |
| **Step 5**: Apply the Quotient Rule. In this case, the x’s ended up in the denominator because there were 10 more x’s in the denominator. | Step 5 |
| **Step 6**: Raise each coefficient (or number) to the appropriate power and then simplify or reduce any remaining fractions. In this case, the fraction does not reduce. | Step 6 |

**Example 2**–Simplify: Example 2

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| **Step 1**: Apply the Zero-Exponent Rule. | Step 1 |
| **Step 2**: Apply the Power Rule. | Step 2 |
| **Step 3**: Apply the Negative Exponent Rule. Move every negative exponent in the numerator to the denominator and vice versa. | Step 3 |
| **Step 4**: Apply the Product Rule. In this case, the product rule does not apply. | Step 4 |
| **Step 5**: Apply the Quotient Rule. In this case, the quotient rule does not apply. | Step 5 |
| **Step 6**: Raise each coefficient (or number) to the appropriate power and then simplify or reduce any remaining fractions. In this case, the fraction does not reduce. | Step 6 |

**Example 3** –Simplify: Example 3

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| **Step 1**: Apply the Zero-Exponent Rule. In this case, there are no zero powers. | Step 1 |
| **Step 2**: Apply the Power Rule. | Step 2 |
| **Step 3**: Apply the Negative Exponent Rule. Move every negative exponent in the numerator to the denominator and vice versa. | Step 3 |
| **Step 4**: Apply the Product Rule. In this case, the product rule does not apply. | Step 4 |
| **Step 5**: Apply the Quotient Rule. In this case, the x’s ended up in the numerator and the y’s ended up in the denominator. | Step 5 |
| **Step 6**: Raise each coefficient (or number) to the appropriate power and then simplify or reduce any remaining fractions. In this case, the fraction does not reduce. | Step 6 |

**Example 4**–Simplify: Example 4

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| **Step 1**: Apply the Zero-Exponent Rule. In this case, after applying the zero-exponent rule and multiplying by 1, that term is essentially gone. | Step 1 |
| **Step 2**: Apply the Power Rule. In this case, I kept the –2 in parentheses because I did not want to lose the negative sign. | Step 2 |
| **Step 3**: Apply the Negative Exponent Rule. Move every negative exponent in the numerator to the denominator and vice versa. | Step 3 |
| **Step 4**: Apply the Product Rule. | Step 4 |
| **Step 5**: Apply the Quotient Rule. In this case, the x’s ended up in the denominator. | Step 5 |
| **Step 6**: Raise each coefficient (or number) to the appropriate power and then simplify or reduce any remaining fractions. In this case, the fraction does reduce. | Step 6 |

Basic rules for logarithms

Since taking a logarithm is the opposite of exponentiation (more precisely, the logarithmic function logbxlogb⁡x is the [inverse function](https://mathinsight.org/definition/inverse_function) of the [exponential function](https://mathinsight.org/exponential_function) bxbx), we can derive the basic rules for logarithms from the [basic rules for exponents](https://mathinsight.org/exponentiation_basic_rules).

For simplicity, we'll write the rules in terms of the natural logarithm ln(x)ln⁡(x). The rules apply for any logarithm logbxlogb⁡x, except that you have to replace any occurence of ee with the new base bb.

The natural log was defined by equations [(1)](https://mathinsight.org/logarithm_basics#mjx-eqn-naturalloga)(1) and [(2)](https://mathinsight.org/logarithm_basics#mjx-eqn-naturallogb)(2). If we plug the value of kk from equation [(1)](https://mathinsight.org/logarithm_basics#mjx-eqn-naturalloga)(1) into equation [(2)](https://mathinsight.org/logarithm_basics#mjx-eqn-naturallogb)(2), we determine that a relationship between the natural log and the exponential function is

elnc=c.(3)(3)eln⁡c=c.

Or, if we plug in the value of cc from [(2)](https://mathinsight.org/logarithm_basics#mjx-eqn-naturallogb)(2) into equation [(1)](https://mathinsight.org/logarithm_basics#mjx-eqn-naturalloga)(1), we'll obtain another relationship

ln(ek)=k.(4)(4)ln⁡(ek)=k.

These equations simply state that exex and lnxln⁡x are inverse functions. We'll use equations [(3)](https://mathinsight.org/logarithm_basics#mjx-eqn-lnexpinversesa)(3) and [(4)](https://mathinsight.org/logarithm_basics#mjx-eqn-lnexpinversesb)(4) to derive the following rules for the logarithm.

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| **Rule or special case** | **Formula** |
| [Product](https://mathinsight.org/logarithm_basics#product) | ln(xy)=ln(x)+ln(y)ln⁡(xy)=ln⁡(x)+ln⁡(y) |
| [Quotient](https://mathinsight.org/logarithm_basics#quotient) | ln(x/y)=ln(x)−ln(y)ln⁡(x/y)=ln⁡(x)−ln⁡(y) |
| [Log of power](https://mathinsight.org/logarithm_basics#log_power) | ln(xy)=yln(x)ln⁡(xy)=yln⁡(x) |
| [Log of ee](https://mathinsight.org/logarithm_basics#log_e) | ln(e)=1ln⁡(e)=1 |
| [Log of one](https://mathinsight.org/logarithm_basics#log_one) | ln(1)=0ln⁡(1)=0 |
| [Log reciprocal](https://mathinsight.org/logarithm_basics#log_reciprocal) | ln(1/x)=−ln(x)ln⁡(1/x)=−ln⁡(x) |

The product rule

We can use the [product rule for exponentiation](https://mathinsight.org/exponentiation_basic_rules#product) to derive a corresponding product rule for logarithms. Using the base b=eb=e, the product rule for exponentials is

eaeb=ea+beaeb=ea+b

for any numbers aa and bb. Starting with the log of the product of xx and yy, ln(xy)ln⁡(xy), we'll use equation [(3)](https://mathinsight.org/logarithm_basics#mjx-eqn-lnexpinversesa)(3) (with c=xyc=xy) to write

eln(xy)=xy.eln⁡(xy)=xy.

Then, we'll use equation [(3)](https://mathinsight.org/logarithm_basics#mjx-eqn-lnexpinversesa)(3) two more times (with c=xc=x and with c=yc=y) to write xyxy in terms of ln(x)ln⁡(x) and ln(y)ln⁡(y),

eln(xy)=xy=eln(x)eln(y).eln⁡(xy)=xy=eln⁡(x)eln⁡(y).

Lastly, we use the product rule for exponents with a=ln(x)a=ln⁡(x) and b=ln(y)b=ln⁡(y) to conclude that

eln(xy)=eln(x)eln(y)=eln(x)+ln(y).eln⁡(xy)=eln⁡(x)eln⁡(y)=eln⁡(x)+ln⁡(y).

When we take the logarithm of both sides of eln(xy)=eln(x)+ln(y)eln⁡(xy)=eln⁡(x)+ln⁡(y), we obtain

ln(eln(xy))=ln(eln(x)+ln(y)).ln⁡(eln⁡(xy))=ln⁡(eln⁡(x)+ln⁡(y)).

The logarithms and exponentials cancel each other out (equation [(4)](https://mathinsight.org/logarithm_basics#mjx-eqn-lnexpinversesb)(4)), giving our product rule for logarithms,

ln(xy)=ln(x)+ln(y).ln⁡(xy)=ln⁡(x)+ln⁡(y).

The quotient rule

The quotient rule for logarithms follows from the [quotient rule for exponentiation](https://mathinsight.org/exponentiation_basic_rules#quotient),

eaeb=ea−beaeb=ea−b

in the same way.

Starting with c=x/yc=x/y in equation [(3)](https://mathinsight.org/logarithm_basics#mjx-eqn-lnexpinversesa)(3) and applying it again with c=xc=x and c=yc=y, we can calculate that

eln(x/y)=xy=eln(x)eln(y)=eln(x)−ln(y),eln⁡(x/y)=xy=eln⁡(x)eln⁡(y)=eln⁡(x)−ln⁡(y),

where in the last step we used the quotient rule for exponentation with a=ln(x)a=ln⁡(x) and b=ln(y)b=ln⁡(y). Since eln(x/y)=eln(x)−ln(y)eln⁡(x/y)=eln⁡(x)−ln⁡(y), we can conclude that the quotient rule for logarithms is

ln(x/y)=ln(x)−ln(y).ln⁡(x/y)=ln⁡(x)−ln⁡(y).

(This last step could follow from, for example, taking logarithms of both sides of eln(x/y)=eln(x)−ln(y)eln⁡(x/y)=eln⁡(x)−ln⁡(y) like we did in the last step for the product rule.)

Log of a power

To obtain the rule for the log of a power, we start with the rule for [power of a power](https://mathinsight.org/exponentiation_basic_rules#power_power),

(ea)b=eab.(5)(5)(ea)b=eab.

Starting with c=xyc=xy in equation [(3)](https://mathinsight.org/logarithm_basics#mjx-eqn-lnexpinversesa)(3) and applying it again, this time just once more with c=xc=x, we can calculate that

eln(xy)=xy=(eln(x))y=eyln(x)eln⁡(xy)=xy=(eln⁡(x))y=eyln⁡(x)

where in the last step we used the power of a power rule for a=ln(x)a=ln⁡(x) and b=yb=y. From eln(xy)=eyln(x)eln⁡(xy)=eyln⁡(x), we can conclude that

ln(xy)=yln(x),ln⁡(xy)=yln⁡(x),

which is the rule for the log of a power.

Log of ee

The formula for the log of ee comes from the formula for the [power of one](https://mathinsight.org/exponentiation_basic_rules#power_one),

e1=e.e1=e.

Just take the logarithm of both sides of this equation and use equation [(4)](https://mathinsight.org/logarithm_basics#mjx-eqn-lnexpinversesb)(4) to conclude that

ln(e)=1.ln⁡(e)=1.

Log of one

The formula for the log of one comes from the formula for the [power of zero](https://mathinsight.org/exponentiation_basic_rules#power_zero),

e0=1.e0=1.

Just take the logarithm of both sides of this equation and use equation [(4)](https://mathinsight.org/logarithm_basics#mjx-eqn-lnexpinversesb)(4) to conclude that

ln(1)=0.ln⁡(1)=0.

Log of reciprocal

The rule for the log of a reciprocal follows from the rule for the [power of negative one](https://mathinsight.org/exponentiation_basic_rules#power_negative_one)

x−1=1xx−1=1x

and the above rule for the [log of a power](https://mathinsight.org/logarithm_basics#log_power). Just substitute y=−1y=−1 into the the log of power rule, and you have that

ln(1/x)=−ln(x).